COSIMPLICIAL MODELS FOR THE LIMIT OF THE GOODWILLIE TOWER

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Abstract. We call attention to the intermediate constructions T_nF in Goodwillie's Calculus of homotopy functors, giving a new model which naturally gives rise to a family of towers filtering the Taylor Tower of a functor. We also establish a surprising equivalence between the homotopy inverse limits of these towers and the homotopy inverse limits of certain cosimplicial resolutions. This equivalence gives a greatly simplified construction for the homotopy inverse limit of the Taylor tower of a functor F under general assumptions.

1. Introduction

Let Δ^n be the n-simplex and $\mathrm{sk}_0\,\Delta^n$ be its 0-skeleton, that is, $n+1$ points. We use $*$ to denote the topological join. For a space X, $\mathrm{sk}_0\,\Delta^0 * X \simeq CX$ and $\mathrm{sk}_0\,\Delta^n * X \simeq \bigvee_n \Sigma X$. Thus, we have the cosimplicial space

$$(\mathrm{sk}_0\,\Delta^* * X) \simeq \; CX \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow} \Sigma X \substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow} \Sigma X \vee \Sigma X \substack{\longrightarrow \\ \vdots \\ \longrightarrow} \cdots$$

Hopkins [Hop84a, Hop84b], and later, Goerss [Goe93], analyzed the spectral sequence associated to this cosimplicial space and showed that when X is connected, it converges to $\mathbb{Z}_\infty X$, the Bousfield $\mathbb{Z}$-nilpotent completion of a space (for details of the construction and applications, see [BK72]).

One result of this paper is a new conceptual proof of this theorem. We assume for the moment that X is a connected space. $\mathrm{P}_\infty\mathbb{I}(X)$ is the inverse limit of the Goodwillie Taylor tower of the identity functor, applied to X. We show the following weak equivalence, where $\mathbb{I}$ is the identity functor of spaces:

$$\operatorname{holim} \mathrm{sk}_0\,\Delta^* * X \sim \mathrm{P}_\infty\mathbb{I}(X).$$

By work of Arone-Kankaanrinta [AK98], $\mathrm{P}_\infty\mathbb{I}(X) \sim \mathbb{Z}_\infty(X)$. This gives the result of Goerss and of Hopkins.

Indeed, we show that $\forall k \geq 0$ and X connected, that $\operatorname{holim} \mathrm{sk}_k\,\Delta^* * X \sim \mathrm{P}_\infty\mathbb{I}(X)$.

More generally, if F is ρ-analytic (see Section 2 for a definition) and X is at least m connected for $m \geq -1$, then we have weak equivalences,

$$\mathrm{P}_\infty F(X) \sim \operatorname{holim}_\Delta F(\mathrm{sk}_k\,\Delta^* * X),$$

for all $k \geq \rho - m$ if $\rho \geq m$, and otherwise it is true for all $k \geq 0$. This arises as a natural corollary of our main results (specifically Corollary 1.4, following from Theorems 1.2 and 1.1 described below).

In order to precisely state our main results, we require some additional definitions and notation.

Let F be a functor from spaces to spaces (or to spectra) which preserves weak equivalences. We will also assume that F commutes with filtered colimits. Goodwillie, in [Goo90, Goo91, Goo03], constructs for such F a tower of functors which approximate $F(X)$ under mild conditions on F and its input, X. Each finite stage of the tower is denoted P_nF, in analogy with the nth partial sums of a Taylor series of a function, which are degree n polynomials. A full definition of a degree n polynomial functor is given in Section 2. The homotopy inverse limit of the Taylor tower is denoted $\mathrm{P}_\infty F$.

In general, to build each of the P_nF's requires taking the homotopy colimit over a directed system. Each finite stage of this system is the iteration of a homotopy inverse limit construction, called T_nF.

Let $\mathscr{P}([n])$ be the powerset on $[n] \in \Delta$, a poset. Posets may be viewed naturally as categories with maps given by the partial order, i.e. inclusion of subsets. We denote by $\mathscr{P}_0([n])$ the full subcategory omitting the empty set.

We then define $\mathrm{T}_nF(X) := \mathrm{holim}_{U\in\mathscr{P}_0([n])} F(U * X)$ and observe that this construction gives us a natural transformation $t_n : F(X) \to \mathrm{T}_nF(X)$, since $F(X) = F(\emptyset * X)$ and the inclusion of the empty set into each U (viewed as an element of $\mathscr{P}([n])$) is compatible with the other maps in the diagram. The t_n's give rise to the maps in the directed system used to construct $\mathrm{P}_nF(X)$:

$$\mathrm{P}_nF(X) := \mathrm{hocolim}_k(\mathrm{T}_nF(X) \xrightarrow{t_n} \mathrm{T}_n^2F(X) \xrightarrow{t_n} \cdots \xrightarrow{t_n} \mathrm{T}_n^kF(X) \xrightarrow{t_n} \cdots)$$

We first establish a new model for each iterated approximation, T_n^kF. Using this model, we obtain maps $\tau^k : \mathrm{T}_{n+1}^kF(X) \to \mathrm{T}_n^kF(X)$, and therefore a tower of partial approximations for each k:

$$\cdots \mathrm{T}_n^kF(X) \xrightarrow{\tau^k} \mathrm{T}_{n-1}^kF(X) \xrightarrow{\tau^k} \cdots \mathrm{T}_1^kF(X)$$

We will need a few definitions, and then may give our model for the T_n^kF's, which will be used in the proof of our main theorem.

Let Δ be the category of finite ordered sets and monotone maps, with objects denoted $[j] = \{0, \dots, j\}$ with the usual order. Let $\Delta_{\leq n}$ be the full subcategory of Δ with objects $[j]$ such that $j \leq n$. We use diag to denote the diagonal of a k-cosimplicial space.

The n-coskeleton of $X^\bullet$, denoted $\mathrm{cosk}_n X^\bullet$, is constructed by precomposing with the inclusion $\Delta_{\leq n} \hookrightarrow \Delta$ and then taking the right Kan extension along the inclusion of the subcategory. We let $\mathrm{cosk}_{\vec{n}}(X^{\vec{\bullet}})$ denote the k-cosimplicial analog, n-coskeleton taken in every dimension.

Theorem 1.1. *For all $k, n \geq 0$, we have the following weak equivalence*

$$\mathrm{T}_n^kF(X) \sim \mathrm{holim}_{\Delta\leq nk}\, \mathrm{diag}(\mathrm{cosk}_{\vec{n}}\, F((\mathrm{sk}_0\, \Delta^\bullet)^{*k} * X))$$

In particular, as $n \to \infty$, we have as an immediate consequence the following equivalence:

$$\mathrm{holim}_n\, \mathrm{T}_n^kF(X) \sim \mathrm{holim}_\Delta\, \mathrm{diag}F((\mathrm{sk}_0\, \Delta^\bullet)^{*k} * X)).$$

We also show that

Theorem 1.2. *For all $k \geq 0$, the functors $\mathrm{sk}_k\, \Delta^\bullet$ and $(\mathrm{sk}_0\, \Delta^\bullet)^{*(k+1)}$ are both homotopy left cofinal as functors from Δ to $(k-1)$-connected spaces of CW type; in particular, for all nonempty spaces X and homotopy endofunctors F, we have weak equivalences*

$$\mathrm{holim}_\Delta\, F(\mathrm{sk}_k\, \Delta^\bullet * X) \sim \mathrm{holim}_\Delta\, F((\mathrm{sk}_0\, \Delta^\bullet)^{*(k+1)} * X)$$

Furthermore, with Theorem 1.1, we have weak equivalences for all $k \geq 0$

$$\mathrm{holim}_\Delta\, F(\mathrm{sk}_k\, \Delta^\bullet * X) \sim \mathrm{holim}_n(\cdots \to \mathrm{T}_n^{k+1}F(X) \to \mathrm{T}_{n-1}^{k+1}F(X) \to \cdots \mathrm{T}_1^{k+1}F(X))$$

The weak equivalences in Theorems 1.1 and 1.2 are natural in k. We discuss the definition and main property of left cofinal functors in Section 2.

Notice that the tower in Theorem 1.2 is over the maps $\tau^{k+1} : \mathrm{T}_n^{k+1}F \to \mathrm{T}_{n-1}^{k+1}F$, along the same stage of iteration of *different* T_nF's. This is markedly different than the directed system used to construct the P_nF's, which is over the maps $t_n : \mathrm{T}_n^jF \to \mathrm{T}_n^{j+1}F$, i.e. along iterations of the *same* T_n construction. We depict both collections of maps in Figure 1. P_nF is then the homotopy colimit along the nth column of Figure 1, whereas the partial approximation towers in Theorem 1.2 are the rows.

Theorem 1.2 provides an equivalence between the homotopy limit over the kth row of the diagram in Figure 1 and the homotopy limit of F applied to $(\mathrm{sk}_k\, \Delta^* * X)$. We will discuss the maps involved in these towers in Section 2.

We now present several consequences of Theorem 1.2.

Corollary 1.3. [1] *For a given j, there are weak equivalences for all $k \geq 0$*

$$\mathrm{P}_jF(X) \sim \mathrm{holim}_\Delta\, \mathrm{P}_jF(\mathrm{sk}_k\, \Delta^\bullet * X)$$

In particular, this also implies that $P_\infty F(X) \sim \mathrm{holim}_\Delta\, P_\infty F(\mathrm{sk}_k\, \Delta^\bullet * X)$.

[1] This can be seen as the unstable extension of a stable pro-result [BEJM11, Prop 4.2]

$$
\begin{array}{rccccccc|c}
\mathrm{P}_\infty F := \mathrm{holim} & (\cdots \longrightarrow & \mathrm{P}_n F & \longrightarrow & \mathrm{P}_{n-1}F & \longrightarrow & & \longrightarrow \ \mathrm{P}_1 F) & \\
 & & \rotatebox{-90}{$\simeq \mathrm{hocolim}\ ($} & & \rotatebox{-90}{$\simeq \mathrm{hocolim}\ ($} & & & \rotatebox{-90}{$\simeq \mathrm{hocolim}\ ($} & \\
 & & \uparrow & & \uparrow & & & \uparrow & \\
 & & \vdots & & \vdots & & & \vdots & \\
 & & \uparrow & & \uparrow & & & \uparrow & \text{row:} \\
 & \cdots \longrightarrow & \mathrm{T}^2_n F & \longrightarrow & \mathrm{T}^2_{n-1} F & \longrightarrow & \cdots & \longrightarrow \ \mathrm{T}^2_1 F & 1 \\
 & & \uparrow & & \uparrow & & & \uparrow & \\
 & \cdots \longrightarrow & \underbrace{\mathrm{T}_n F} & \longrightarrow & \underbrace{\mathrm{T}_{n-1} F} & \longrightarrow & \cdots & \longrightarrow \ \underbrace{\mathrm{T}_1 F} & 0 \\
\hline
 & \text{column:} & n & & (n-1) & & & 1 &
\end{array}
$$

FIGURE 1. Partial approximations

A functor F is said to be ρ-analytic if its failure to be polynomial of any degree is bounded in a way that depends on ρ. One important consequence of ρ-analyticity is that for X at least ρ-connected, $F(X) \simeq \mathrm{P}_\infty F(X)$. This is the consequence of analyticity which we need for our purposes; for a thorough definition of analyticity and discussion of its implications, see [Goo91].

We will use this fact and the preceding corollary to establish the following:

Corollary 1.4. *Let F be a ρ-analytic functor. Then we have weak equivalences $\forall k > \rho$,*

$$\mathrm{P}_\infty F(X) \sim \mathrm{holim}_n(\cdots \mathrm{T}^{k+1}_n F(X) \xrightarrow{\tau^{k+1}} \mathrm{T}^{k+1}_{n-1} F(X) \xrightarrow{\tau^{k+1}} \cdots \xrightarrow{\tau^{k+1}} \mathrm{T}^{k+1}_1 F(X)) \sim \mathrm{holim}_\Delta F(\mathrm{sk}_k \Delta^* * X)$$

If we raise the connectivity of X (denoted conn(X)), we may improve this to all $k \geq max(\rho - conn(X) - 1, 0)$.

Proof. For two spaces, X and Y, the connectivity of their join, $X*Y$, is equal to $\mathrm{conn}(X)+\mathrm{conn}(Y)+2$. For each m, $\mathrm{sk}_k \Delta^m$ is $(k-1)$-connected, so $\mathrm{conn}(\mathrm{sk}_k \Delta^m * X) = (k+1) + \mathrm{conn}(X)$. We conclude that the connectivity of the cosimplicial space $\mathrm{sk}_k \Delta^\bullet * X$ is the same. In particular, X arbitrary (of connectivity ≥ -2), $\mathrm{sk}_k \Delta^\bullet * X$ has connectivity at least $(k-1)$ and for F of analyticity ρ such that $\rho - 1 \leq k$, $F(\mathrm{sk}_k \Delta^\bullet * X) \simeq \mathrm{P}_\infty F(\mathrm{sk}_k \Delta^\bullet * X)$.

We apply holim to this equality, and also recall that by Cor 1.3, $\mathrm{P}_\infty F(X) \sim \mathrm{holim}_\Delta \mathrm{P}_\infty F(\mathrm{sk}_k \Delta^\bullet * X)$. The resulting equivalence is $\mathrm{P}_\infty F(X) \sim \mathrm{holim}_\Delta F(\mathrm{sk}_k \Delta^\bullet * X)$. We follow with Theorem 1.2 and obtain the other stated equivalence. □

We note that this corollary yields $\mathrm{P}_\infty F(X)$ via a construction which no longer makes use of any *infinite* colimits, and that each T^k_n commutes with holim and hofiber.

If F is ρ-analytic, and X is in its radius of convergence (i.e. at least ρ-connected), then Cor 1.4 implies

$$F(X) \sim \mathrm{P}_\infty F(X) \sim \mathrm{holim}_\Delta F(\mathrm{sk}_0 \Delta^\bullet * X).$$

This result may be rephrased as saying that for F an analytic functor, and a space X in its "radius of convergence", $F(X)$ is well approximated by $F(- * X)$ applied to finite nonempty sets.

By a similar proof to that showing that the identity is 1-analytic[2], it follows that if a functor F commutes with realizations and preserves filtered colimits, then it is 1-analytic. Corollary 1.4 then gives us the following:

Corollary 1.5. *If F commutes with realizations and preserves filtered colimits, the equivalence*

$$\mathrm{P}_\infty F(X) \sim \mathrm{holim}_n(\cdots \mathrm{T}^{k+1}_n F(X) \xrightarrow{\tau^{k+1}} \mathrm{T}^{k+1}_{n-1} F(X) \xrightarrow{\tau^{k+1}} \cdots \mathrm{T}^{k+1}_1 F(X)).$$

holds for all $k > 1$.

[2]For the identity functor, this may be found in [Goo91, Example 4.3, Theorem 2.3]

Proposition 1.6. *For the identity $\mathbb{I}$ from spaces to spaces, X a connected space, and $\mathbb{Z}_\infty X$ the Bousfield $\mathbb{Z}$-nilpotent completion of X, we have that for all $k \geq 0$, the following weak equivalence*

$$\mathrm{holim}_\Delta(\mathrm{sk}_k\,\Delta^\bullet * X) \sim \mathrm{P}_\infty(\mathbb{I})(X) \sim \mathbb{Z}_\infty X,$$

and when X is already nilpotent, $\mathrm{holim}_\Delta(\mathrm{sk}_k\,\Delta^\bullet * X) \sim X$.

Proof. Since $\mathbb{I}$ is 1-analytic, Corollary 1.4 allows us to conclude that for any space X, $\mathrm{holim}_\Delta(\mathrm{sk}_k\,\Delta^\bullet * X) \sim \mathrm{P}_\infty(\mathbb{I})(X)$ for all $k \geq 2$. Restricting to X which are 0-connected, i.e. raising the minimum connectivity from -2 to 0, changes this equivalence to hold for all $k \geq 0$. Then, by Arone-Kankaanrinta [AK98, §3], we have that for X connected, $\mathrm{P}_\infty(\mathbb{I})(X) \sim \mathbb{Z}_\infty X$, and for X nilpotent, $\mathrm{P}_\infty(\mathbb{I})(X) \sim X$. □

If, instead of combining our result with that of Arone-Kankaanrinta, we combine with the result of Goerss and Hopkins, one can view our main result as justification for why the spectral sequences associated to the Taylor Tower of the identity of spaces and that associated to the $\mathbb{Z}$-nilpotent completion of a space abut to the same thing [3].

Remark 1.7. *In highlighting these partial approximation towers, we also raise the question of what being weakly equivalent to $\mathrm{P}_\infty F(X)$ off of the radius of convergence is telling us. This should in general provide conditions that give the "true" radius of convergence of a functor. That is, given F a ρ-analytic functor, it also converges for spaces X such that $F(X) \sim \mathrm{holim}_\Delta F(\mathrm{sk}_k\,\Delta^\bullet * X)$ for all $k \geq min(\rho - m - 1, 0)$, where X is at least m-connected.*

We also suspect that the kth layer is a sort of k-analytic approximation of a ρ-analytic functor, where k is less than ρ.

Note on translating between the language of Goerss [Goe93] *and our current terminology:*
Goerss defines a cosimplicial construction $C(X, X)$ for a nonempty space X which sends $[n]$ to $\bigvee_n \Sigma X$ (taking the empty wedge here as CX). As mentioned earlier, this is equivalent to $(\mathrm{sk}_0\,\Delta^* * X)$. Theorem 1.1 in [Goe93] implies that when X is connected, $\mathrm{holim}_\Delta C(X, X) \sim \mathbb{Z}_\infty X$, that is, $\mathrm{holim}_n \mathrm{T}_n\mathbb{I}(X) := \mathrm{holim}_\Delta(\mathrm{sk}_0\,\Delta^* * X) \sim \mathbb{Z}_\infty X$.

Note on translating between the language of Hopkins and our current terminology:
This is from [Hop84a], Section 3, p221-222. He lets C_n be what we call $\mathscr{P}_0([n])$. He defines, for a given space X, a functor F^n, as the homotopy inverse limit of a (punctured) cube. For $A \in \mathscr{P}_0([n]) =: C_n$, the A-indexed position of this $(n+1)$-cube is the homotopy colimit of X mapping to $|A|$ different copies of a point, which we will explain shortly. He denotes this by F^nA. Regarding A as a finite ordered set, we can view F^nA as the homotopy pushout of the following:

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \arrow[drr] & & \\ \{0\} & \{1\} & \cdots & \{|A|-1\} \end{tikzcd}$$

We replace these maps by cofibrations (since we are taking a homotopy colimit), giving us that we are pushing out over the following diagram:

$$\begin{tikzcd} & X \arrow[dl] \arrow[d] \arrow[drr] & & \\ \{0\} * X & \{1\} * X & \cdots & \{|A|-1\} * X \end{tikzcd}$$

That is, the A-indexed position of this $(n+1)$ cube is $A * X$.

Then $F^n := \mathrm{holim}_{A \in \mathscr{P}_0([n])} F^nA \simeq \mathrm{holim}_{U \mathscr{P}_0([n])} U * X$. That is, we have shown that his F^n's exactly the $\mathrm{T}_n\mathbb{I}(X)$'s. He constructs a tower of these F^n's:

$$(\cdots \to \mathrm{holim}\, F^n \to \mathrm{holim}\, F^{n-1} \to \cdots \mathrm{holim}\, F^1)$$

[3]The spectral sequence associated to the Taylor Tower of $\mathbb{I}(X)$ takes as input the collection of $D_n\mathbb{I}(X)$, which were computed by Johnson [Joh95].

which is therefore our $\mathrm{T}_n\mathbb{I}$ tower,

$$(\cdots \to \mathrm{T}_n\mathbb{I}(X) \to \mathrm{T}_{n-1}\mathbb{I}(X) \to \cdots \mathrm{T}_1\mathbb{I}(X)).$$

Theorem 3.2.2 of [Hop84b] is that the homotopy inverse limit of a construction that is equivalent to the tower of F^n's gives $\mathbb{Z}_\infty X$ when X is connected, i.e. that $\mathrm{holim}_n \mathrm{T}_n\mathbb{I}(X) \sim \mathbb{Z}_\infty X$.

1.1. **Organization.** The remainder of this paper is organized as follows. Section 2 gives background mainly on the Calculus. Theorem 1.1 , the more geometric and cosimplicial interpretation of T_nF, is proven in Section 3. The proof of Theorem 1.2 is then given in Section 4.

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2. Background

We will first discuss cofinality, and then review the required constructions of Goodwillie Calculus. For this paper, we restrict our attention to functors F from spaces to spaces (not necessarily based) which commute with filtered colimits. F is a *homotopy functor* if it preserves weak equivalences.

2.1. **Homotopy left cofinality.** This is the definition of homotopy left cofinal which we will use. There are also corresponding notions of non-homotopy cofinality (involving strict limits) and a dual notion of right cofinality (involving colimits) which we will not discuss.

Definition 2.1 (see [Hir02], Definition 19.6.1 p418)**.** *Let $\mathscr{D}$ be a small category and for all A, B objects of $\mathscr{D}$, we denote by $Mor_{\mathscr{D}}(A, B)$ the set of morphisms in $\mathscr{D}$ between them. Let $G : \Delta \to \mathscr{D}$. The functor G is* ***homotopy left cofinal*** *if for every object α of $\mathscr{D}$, the simplicial set $n \mapsto Mor_{\mathscr{D}}(G(n), \alpha)$ is contractible.*

The following consequence of being homotopy left cofinal is what we use to establish our equivalences:

Lemma 2.2 (see [Hir02], Theorem 19.6.7 & 16.6.23)**.** *Let M be a simplicial model category, let $\mathscr{C}, \mathscr{D}$ be small categories, and let $G : \mathscr{C} \to \mathscr{D}$ be a functor. If G is* ***homotopy left cofinal****, then for every object-wise fibrant $\mathscr{D}$-diagram F in M, we have that the following natural map of homotopy limits is a weak equivalence:*

$$\mathrm{holim}_{\mathscr{D}} F \to \mathrm{holim}_{\mathscr{C}} F \circ G$$

We will be working only with a simplicial model category M where the objects are all fibrant, namely, the category of spaces with cofibrations the cellular inclusions of CW complexes, so object-wise fibrancy comes for free.

Next, we will review the Calculus.

2.2. **Excisive functors.** In [Goo90], Goodwillie establishes the following condition for a functor, which is in analogy with a function being polynomial of degree 1:

Definition 2.3. *A functor is 1-excisive if F takes homotopy pushout (called cocartesian) squares to homotopy pullback (called cartesian) squares.*

This may not be the most familiar statement of excision, compared to its usual statement as one of the axioms of a (generalized) homology theory (as in [ES52]) . There is a nice discussion in [MO02, p.22] of how to get from excision as usually stated in the Eilenberg-Steenrod axioms to this definition.

2.3. **Excisive approximation.** The following is a pushout of finite sets. It is also a diagrammatic representation of the category $\mathscr{P}([1])$.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \{0\} \arrow[d] \\ \{1\} \arrow[r] & \{0,1\} \end{tikzcd}$$

We make the following definition:

$$\mathrm{T}_1F(X) := \mathrm{holim}_{U\in\mathscr{P}_0([1])} F(U * X)$$

As a result, we have a natural transformation $t_1 : F(X) \to \mathrm{T}_1F(X)$, induced by the natural map:

$$F(X) = F(\emptyset * X) \to \underset{U\in\mathscr{P}_0([1])}{\mathrm{holim}} (U \mapsto F(U * X)).$$

That is, the map from the initial object of the square, $F(X)$ to the homotopy pullback of the rest, $\mathrm{T}_1F(X)$. We can take T_1 of T_1F, and also have the same natural transformation from initial to homotopy pullback, now $t_1 : \mathrm{T}_1F(X) \to \mathrm{T}_1(\mathrm{T}_1F(X)) =: \mathrm{T}_1^2F(X)$. See Figure 2.

We define the 1-excisive approximation to F, P_1F, as the following homotopy colimit over iterations:

$$\mathrm{P}_1F(X) := \mathrm{hocolim}(\mathrm{T}_1F(X) \xrightarrow{t_1} \mathrm{T}_1^2F(X) \xrightarrow{t_1} \cdots)$$

$$\mathrm{T}_1^2F(X) \quad := \quad \mathrm{holim}\left(\begin{tikzcd} & \mathrm{T}_1F(\{0\}*X) \arrow[d] \\ \mathrm{T}_1F(\{1\}*X) \arrow[r] & \mathrm{T}_1F(\{0,1\}*X) \end{tikzcd} \right)$$

$$\simeq \quad \mathrm{holim}\left(\begin{array}{ccc} & & \left(\begin{tikzcd} & F(\{0\}*\{0\}*X) \arrow[d] \\ F(\{0\}*\{1\}*X) \arrow[r] & F(\{0\}*\{0,1\}*X) \end{tikzcd} \right) \\ & & \downarrow \\ \left(\begin{tikzcd} & F(\{1\}*\{0\}*X) \arrow[d] \\ F(\{1\}*\{1\}*X) \arrow[r] & F(\{1\}*\{0,1\}*X) \end{tikzcd} \right) & \to & \left(\begin{tikzcd} & F(\{0,1\}*\{0\}*X) \arrow[d] \\ F(\{0,1\}*\{1\}*X) \arrow[r] & F(\{0,1\}*\{0,1\}*X) \end{tikzcd} \right) \end{array} \right)$$

FIGURE 2. $\mathrm{T}_1^2F(X)$

2.4. **Higher Degree Functors.** As in the 1-excisive case, we begin with a diagrammatical representation of the powerset category, now $\mathscr{P}([n])$, which is an $(n+1)$-cube indexed by subsets of $[n]$.

Definition 2.4. *We say that a $\mathscr{P}([n])$-indexed diagram (i.e. an $(n+1)$-cube) $\mathscr{X}$ is* <u>*strongly co-cartesian*</u> *if every square (i.e. 2-dimensional) sub-face is cocartesian.*

Definition 2.5. *We say that a functor F is* <u>*n-excisive*</u> *if it takes strongly co-cartesian $(n+1)$-cubes to cartesian $(n+1)$ cubes.*

Analogous to the 1-excisive case, we make the following definition:

$$\mathrm{T}_nF(X) := \mathrm{holim}_{U\in\mathscr{P}_0([n])} F(U * X)$$

This allows us to express $t_n : F(X) \to \mathrm{T}_nF(X)$ as the natural map:

$$F(X) = F(\emptyset * X) \to \underset{U\in\mathscr{P}_0([n])}{\mathrm{holim}} (U \mapsto F(U * X)).$$

As before, we define the degree n polynomial approximation to F, P_nF, as the following homotopy colimit:

$$\mathrm{P}_nF(X) := \mathrm{hocolim}(\mathrm{T}_nF(X) \xrightarrow{t_n} \mathrm{T}_n^2F(X) \xrightarrow{t_n} \cdots)$$

2.5. **Taylor Tower.** The collection of polynomial approximations to a functor F, $\{\mathrm{P}_nF\}_{n\geq 0}$, come with natural maps $\mathrm{P}_nF(X) \to \mathrm{P}_{n-1}F(X)$ for all $n \geq 1$. Using Goodwillie's [Goo03] definition

$$(\mathrm{T}_n^iF)(X) := \operatorname*{holim}_{(U_1,\dots,U_i)\in\mathscr{P}_0([n+1])^i} F(X * (U_1 * \cdots * U_i))$$

we then have for all $i, n \geq 1$ a natural map $\mathrm{T}_n^iF \xrightarrow{q_{n,i}} \mathrm{T}_{n-1}^iF$ induced by the inclusion of categories, $\mathscr{P}_0([n])^i \hookrightarrow \mathscr{P}_0([n+1])^i$. Taking the colimit along i gives us the induced map $\mathrm{P}_nF \xrightarrow{q_n} \mathrm{P}_{n-1}F$. With these maps we form a tower, the Goodwillie Taylor Tower of $F(X)$:

$$\cdots \to \mathrm{P}_nF(X) \xrightarrow{q_n} \mathrm{P}_{n-1}F(X) \xrightarrow{q_{n-1}} \cdots \longrightarrow \mathrm{P}_1F(X) \xrightarrow{q_1} \mathrm{P}_0F(X)$$

We denote by $\mathrm{P}_\infty F(X)$ the homotopy inverse limit of this tower.

This defines $\mathrm{P}_\infty F(X)$ as the homotopy inverse limit of a collection of constructions which are themselves homotopy colimits (of finite homotopy inverse limits). That is, it is not expected that this construction will commonly commute with either colimits or limits; there are several special cases set out in [Goo03].

2.6. **Analyticity.** Let ρ be an integer greater than or equal to zero. We say that a functor F is *ρ-analytic* if its failure to be n-excisive is controlled by ρ as n increases. A precise definition of ρ-analyticity may be found in [Goo91]. We need only the following consequence of being ρ-analytic.

For a ρ-analytic functor, F, and any ρ-connected space, X, the following natural map is an equivalence $F(X) \xrightarrow{\sim} \mathrm{P}_\infty F(X)$.

Note: Higher values of ρ mean that X is 'closer' to 0 (i.e. $*$), since increasing connectivity means that X has more vanishing homotopy groups. A *lower* value of ρ means a larger "radius of convergence" of the functor F.

3. Proof of Theorem 1.1

In this section, we prove the following theorem:

Theorem 1.1 *For all $k, n \geq 0$, we have the following weak equivalence*

$$\mathrm{T}_n^kF(X) \sim \mathrm{holim}_{\Delta_{\leq nk}}\, \mathrm{diag}(\mathrm{cosk}_{\vec{n}}\, F((\mathrm{sk}_0\, \Delta^\bullet)^{*k} * X))$$

In particular, as $n \to \infty$, we have as an immediate consequence the following equivalence:

$$\mathrm{holim}_n\, \mathrm{T}_n^kF(X) \sim \mathrm{holim}_\Delta\, \mathrm{diag}F((\mathrm{sk}_0\, \Delta^\bullet)^{*k} * X)).$$

The heart of this proof is establishing a finite-dimensional analog for the generalized n-cosimplicial Eilenberg-Zilber Theorem (see [Shi96, Prop 8.1]).

First, we make use of a lemma to switch models for T_nF.

Lemma 3.1 ([Sin09] Theorem 6.7, or [Hop84b], §3.1 Prop 3.1.4)**.** *Let $c_n : \mathscr{P}_\emptyset([n]) \to \Delta_{\leq n}$ be the functor which sends a nonempty subset subset S to $[\#S-1]$ and which sends an inclusion $S \subseteq S'$ to the composite $[\#S-1] \cong S \subset S' \cong [\#S'-1]$. c_n is homotopy left cofinal.*

The immediate consequence of this lemma (plus Lemma 2.2, which outlines the relevant consequence of cofinality) is that we can move between the two models for each T_nF, i.e. the following are weakly equivalent:

$$\mathrm{holim}_{\Delta_{\leq n}}\, F(\mathrm{sk}_0\, \Delta^\bullet * X) \xleftarrow{\sim} \mathrm{holim}_{U\subset\mathscr{P}_0([n])}\, F(U * X) =: \mathrm{T}_nF(X)$$

For the next step of the proof, we will need to introduce some new notation. Consider X an n-cosimplicial space and, for various i and j, $\mathrm{cosk}_j^i\, X$ an n-cosimplicial space with the j-coskeleton

taken in the ith direction. This will be explained for bicosimplicial objects, and the constructions generalize to the n-cosimplicial case.

Let $X^{\bullet,\bullet}$ be bicosimplicial, with directions 1 and 2. For each p and each q, we have cosimplicial spaces $X^{p,\bullet}$ and $X^{\bullet,q}$. For each p, we can consider (separately, not necessarily for all p at once) the cosimplicial space $\operatorname{cosk}_j X^{p,\bullet}$ (and similarly for each q). Then, the bicosimplicial space X with j-coskeleton taken in the 1st direction, denoted $\operatorname{cosk}^1_j X$, is the functor $q \mapsto (p \mapsto \operatorname{cosk}_j X^{p,\bullet})^q$.

We also note that these then give rise to partial holims or totalizations taken in the various directions. Let X be an n-cosimplicial space, then the homotopy limit taken only in the ith direction, $\operatorname{holim}^i_\Delta X$ and the jth holim in the ith direction, $\operatorname{holim}^i_{\Delta_{\leq j}} X$, both produce $(n-1)$-cosimplicial spaces.

Proposition 3.2. [4] *Let X be an n-cosimplicial space with the property that $\operatorname{holim}^i_\Delta X$, is equivalent to $\operatorname{holim}^i_{\Delta_{\leq j_i}} X$, for each $i \in \{1, \dots, n\}$ and some $j_i \geq 1$. If X is fibrant, this is the same condition as $\operatorname{Tot}^i X \simeq \operatorname{Tot}^i_{j_i} X$. Let $J = \Sigma_i j_i$. Then*

$$\operatorname{holim}_\Delta \operatorname{diag}(X) \simeq \operatorname{holim}_{\Delta_{\leq J}} \operatorname{diag}(X).$$

That is, for an arbitrary n-cosimplicial space, Y, we have that

$$\operatorname{holim}^1_{\Delta_{\leq j_1}} \operatorname{holim}^2_{\Delta_{\leq j_2}} \cdots \operatorname{holim}^n_{\Delta_{\leq j_n}} Y \simeq \operatorname{holim}_{\Delta \leq J} \operatorname{diag} \operatorname{cosk}_{j_1, j_2, \dots, j_n} Y,$$

i.e. $\operatorname{Tot}^1_{j_1} \operatorname{Tot}^2_{j_2} \cdots \operatorname{Tot}^n_{j_n} Y \simeq \operatorname{Tot}_J \operatorname{diag} \operatorname{cosk}_{j_1, j_2, \dots, j_n} Y$ for Y fibrant. Here, $\operatorname{cosk}_{j_1, j_2, \dots, j_n}$ denotes the j_i-th coskeleton taken in the ith direction for all $i \in \{1, \dots, n\}$.

The notation and method of this proof will be simplified by using Tot instead of holim. Since the first step of taking the homotopy limit is normally to fibrantly replace, we will assume our spaces are fibrant (or already fibrantly replaced) and will treat $\operatorname{Tot} X$ as equivalent to $\operatorname{holim}_\Delta X$. We first establish the bicosimplicial case, and then conclude n-cosimplicial by induction.

Lemma 3.3. *Let X be a bicosimplicial simplicial set. Then*

$$\operatorname{Tot}^1_p \operatorname{Tot}^2_q X = \operatorname{Tot}(\operatorname{cosk}^1_p \operatorname{cosk}^2_q X) \simeq \operatorname{Tot}_{p+q} \operatorname{diag}(\operatorname{cosk}^1_p \operatorname{cosk}^2_q X)$$

If $\operatorname{Tot}^1_p X \simeq \operatorname{Tot}^1 X$ and $\operatorname{Tot}^2_q X \simeq \operatorname{Tot}^2 X$, then

$$\operatorname{Tot}(X) \simeq \operatorname{Tot}_{p+q} \operatorname{diag}(X)$$

Let sSets be the category of simplicial sets, then $sSets^{\Delta\times\Delta}$ is the category of bicosimplicial simplicial sets.

Proof of Lemma 3.3. Recall that $\operatorname{Tot}(X) := \operatorname{Hom}_{sSets^{\Delta\times\Delta}}(\Delta^{s_1}_t \times \Delta^{s_2}_t, X^{s_1,s_2}_t)$ (enriched Hom). Using Proposition 8.1 of [Shi96], this is also equivalent to $\operatorname{Hom}_{sSets^\Delta}(\Delta^s_t, X^{s,s}_t)$. Likewise,

$$\operatorname{Tot}(\operatorname{cosk}^1_p \operatorname{cosk}^2_q X) \cong \operatorname{Hom}_{sSet^\Delta}(\Delta^s_t, (\operatorname{cosk}^1_p \operatorname{cosk}^2_q X)^{s,s}_t). \tag{1}$$

By the Yoneda Lemma applied to $X \in sSets^{\Delta\times\Delta}$,

$$a, b, c \mapsto X^{a,b}_c \cong a, b, c \mapsto \operatorname{Hom}_{Set^{\Delta\times\Delta}}(\Delta^\bullet_a \times \Delta^*_b, X^{\bullet,*}_c)$$
$$a, c \mapsto X^{a,a}_c \cong a, c \mapsto \operatorname{Hom}_{Set^\Delta}(\Delta^\bullet_a, \operatorname{diag}(X^{\bullet,*})_c)$$

Given this, and combined with the isomorphisms $\operatorname{Tot}_n(X) \cong \operatorname{Hom}(\operatorname{sk}_n \Delta, X) \simeq \operatorname{Hom}(\Delta, \operatorname{cosk}_n X)$, we also have

$$a, b, c \mapsto (\operatorname{cosk}^1_p \operatorname{cosk}^2_q X)^{a,b}_c \cong a, b, c \mapsto \operatorname{Hom}_{Set^{\Delta\times\Delta}}((\operatorname{sk}_p \Delta^\bullet)_a \times (\operatorname{sk}_q \Delta^*)_b, X^{\bullet,*}_c).$$

We continue the proof, making use of this in the first of the following equivalences:

[4]With different notation/language (and alternate proof), this is [BEJM11, Lemma 2.9]

$$
\begin{array}{lll}
(1) \cong & \mathrm{Hom}_{sSet^{\Delta}}(\Delta^s_t, \mathrm{Hom}_{Set^{\Delta\times\Delta}}(\mathrm{sk}_p\,\Delta^\bullet_s \times \mathrm{sk}_q\,\Delta^*_s, X^{*,\bullet}_t)) & \\
\cong & \mathrm{Hom}_{sSet^{\Delta\times\Delta}}(\Delta^s_t \otimes_\Delta (\mathrm{sk}_p\,\Delta^\bullet_s \times \mathrm{sk}_q\,\Delta^*_s), X^{*,\bullet}_t) & \mathrm{Hom}_{Set^{\Delta\times\Delta}}(G, -\,) \text{ right adjoint to } -\otimes_\Delta G \\
\cong & \mathrm{Hom}_{sSet^{\Delta\times\Delta}}(\Delta^s_t \otimes_\Delta \mathrm{sk}_{p+q}(\mathrm{sk}_p\,\Delta^\bullet_s \times \mathrm{sk}_q\,\Delta^*_s), X^{*,\bullet}_t) & (\mathrm{sk}_p\,\Delta^\bullet_s \times \mathrm{sk}_q\,\Delta^*_s) \cong \mathrm{sk}_{p+q}(\mathrm{sk}_p\,\Delta^\bullet_s \times \mathrm{sk}_q\,\Delta^*_s) \\
\cong & \mathrm{Hom}_{sSet^{\Delta\times\Delta}}(\mathrm{sk}_{p+q}\,\Delta^s_t \otimes_\Delta (\mathrm{sk}_p\,\Delta^\bullet_s \times \mathrm{sk}_q\,\Delta^*_s), X^{*,\bullet}_t) & \mathrm{sk}_n \text{ is a tensor, } \mathrm{sk}_n\,Y = \Delta \otimes_{\Delta_{\le n}} Y \\
\cong & \mathrm{Hom}_{sSet^{\Delta}}(\mathrm{sk}_{p+q}\,\Delta^s_t, \mathrm{Hom}_{Set^{\Delta\times\Delta}}(\mathrm{sk}_p\,\Delta^\bullet_s \times \mathrm{sk}_q\,\Delta^*_s, X^{*,\bullet}_t) & \mathrm{Hom}, \otimes \text{ adjunction again} \\
\cong & \mathrm{Hom}_{sSet^{\Delta}}(\mathrm{sk}_{p+q}\,\Delta^s_t, (\mathrm{cosk}^1_p\,\mathrm{cosk}^2_q\,X)^{s,s}_t) & \text{Definition/Yoneda} \\
\cong & \mathrm{Tot}_{p+q}\,\mathrm{diag}(\mathrm{cosk}^1_p\,\mathrm{cosk}^2_q\,X) &
\end{array}
$$

We made use of the fact that n-skeleton is a tensor, by means of the following yoga:

$$\Delta^* \otimes \mathrm{sk}_n\,Y = \Delta^* \otimes_\Delta \Delta \otimes_{\Delta_{\le n}} Y = \Delta^* \otimes_{\Delta_{\le n}} Y = \Delta^* \otimes_{\Delta_{\le n}} \otimes_\Delta \Delta \otimes_\Delta Y = sk_n \Delta^* \otimes_\Delta Y$$

□

Proof of Prop3.2. Recall that X is n-cosimplicial and $\mathrm{Tot}^i_{j_i} X \simeq \mathrm{Tot}^i X$ for all $i \in \{1, \ldots, n\}$. Our inductive hypothesis is that

$$\mathrm{Tot}^{n-1}_{j_{n-1}} \cdots \mathrm{Tot}^2_{j_2}\,\mathrm{Tot}^1_{j_1}\,X \simeq \mathrm{Tot}_{\sum_{i=1}^{n-1} j_i}\,\mathrm{diag}X.$$

The diagonal is taken in the directions we are totalizing in, which we will abusively denote as diag$X$; it should be clear from context what is meant.

Now consider $\mathrm{Tot}^n_{j_n}\,\mathrm{Tot}^{n-1}_{j_{n-1}} \cdots \mathrm{Tot}^2_{j_2}\,\mathrm{Tot}^1_{j_1}\,X$.

$$
\begin{array}{rcll}
\mathrm{Tot}^n_{j_n}\,\mathrm{Tot}^{n-1}_{j_{n-1}} \cdots \mathrm{Tot}^2_{j_2}\,\mathrm{Tot}^1_{j_1}\,X & \simeq & \mathrm{Tot}_{j_n}\,\mathrm{Tot}_{\sum_{i=1}^{n-1} j_i}\,\mathrm{diag}X & \text{Inductive Hypothesis} \\
& \simeq & \mathrm{Tot}_{\sum_{i=1}^{n} j_i}\,\mathrm{diag}(\mathrm{diag}X) & \text{Lemma 3.3} \\
& \simeq & \mathrm{Tot}_{\sum_{i=1}^{n} j_i}\,\mathrm{diag}X &
\end{array}
$$

The last step is that taking the diagonal of the entire object is equivalent to taking the diagonal in the first k dimensions and *then* the diagonal of that collapsed out part and the rest.

□

Proof of Theorem 1.1. Now that we have Proposition 3.2, this is nearly immediate.

$$
\begin{array}{rcll}
\mathrm{T}^k_n F(X) & := & \mathrm{T}_n(\mathrm{T}_n(\cdots(\mathrm{T}_n F(X)) & \\
& \sim & \mathrm{holim}_{\Delta_{\le n}} \cdots \mathrm{holim}_{\Delta_{\le n}} F((\mathrm{sk}_0\,\Delta^\bullet)^{*k} * X) & \text{Lemma 3.1} \\
& \sim & \mathrm{holim}_{\Delta_{\le n} \times \cdots \Delta_{\le n}} F((\mathrm{sk}_0\,\Delta^\bullet)^{*k} * X) & \text{Fubini} \\
& \sim & \mathrm{holim}_{\Delta_{\le nk}}\,\mathrm{diag}\,\mathrm{cosk}_{\vec{n}}\,F((\mathrm{sk}_0\,\Delta^\bullet)^{*k} * X) & \text{Prop 3.2}
\end{array}
$$

□

4. Proof of Theorem 1.2

In this section, we will prove the following theorem:

Theorem 1.2 *For all $k \ge 0$, the functors $\mathrm{sk}_k\,\Delta^\bullet$ and $(\mathrm{sk}_0\,\Delta^\bullet)^{*(k+1)}$ are both homotopy left cofinal as functors from Δ to $(k-1)$-connected spaces; in particular, for all spaces X and homotopy endofunctors F, we have weak equivalences*

$$\mathrm{holim}_\Delta\,F(\mathrm{sk}_0\,\Delta^\bullet * X) \sim \mathrm{holim}_\Delta\,F((\mathrm{sk}_0\,\Delta^\bullet)^{*(k+1)} * X)$$

Furthermore, with Theorem 1.1, we have weak equivalences for all $k \geq 0$

$$\operatorname{holim}_{\Delta} F(\operatorname{sk}_k \Delta^{\bullet} * X) \sim \operatorname{holim}_n(\cdots \to \mathrm{T}_n^{k+1} F(X) \to \mathrm{T}_{n-1}^{k+1} F(X) \to \cdots \mathrm{T}_1^{k+1} F(X))$$

We first make a few definitions. Let $\mathcal{K}$ be the category of $(k-1)$-connected spaces of CW type. Define functors $\mathscr{X}_k, \mathscr{Y}_k : \Delta \to \mathcal{K}$ such that

$$\begin{array}{rcl} \mathscr{X}_k(p) & = & \operatorname{sk}_k \Delta^p \\ \mathscr{Y}_k(p) & = & \underbrace{\operatorname{sk}_0 \Delta^p * \cdots * \operatorname{sk}_0 \Delta^p}_{(k+1) \text{ copies}} \end{array}$$

Clearly, when $k = 0$, these two functors are the same. Notice also that for $k = 1$, we have $\mathscr{X}_1(p) = K_p$ (the complete graph on $(p+1)$ vertices) and $\mathscr{Y}_1(p) = K_{p+1,p+1}$ (the complete bipartite graph on two sets of $(p+1)$ vertices). See figure 3.

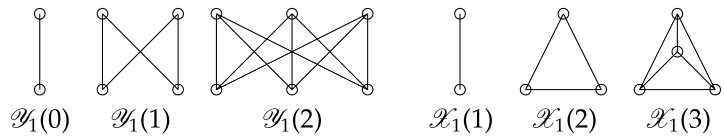

FIGURE 3. $\mathscr{Y}_1$ of 0,1, and 2 and $\mathscr{X}_1$ of 1,2,and 3

To show left cofinality of the two functors, we show that for all $Z \in (k-1)$-connected spaces of CW type, the simplicial sets $p \mapsto \mathrm{Top}_{cts}(\mathscr{X}_k(p), Z)$ and $p \mapsto \mathrm{Top}_{cts}(\mathscr{Y}_k(p), Z)$ are contractible.

Our proofs will make use of the following lemma, which establishes that for $k = 0$, both simplicial sets are contractible:

Lemma 4.1. *Let $\operatorname{sk}_0 \Delta^{\bullet}$ be the cosimplicial space sending $[n]$ to the 0 skeleton of Δ^n, the topological n-simplex. Then for any nonempty space Z, the simplicial set $k \mapsto \mathrm{Top}(\operatorname{sk}_0 \Delta^k, Z)$ is contractible by a contracting homotopy.*

Proof. For each k, $\operatorname{sk}_0 \Delta^k$ is the discrete space with $(k+1)$ points. This allows us to write $\mathrm{Top}(\operatorname{sk}_0 \Delta^k, Z)$ as $\prod^{k+1} Z$. That is, our simplicial set is of the form

$$Z \leftrightarrows Z \times Z \leftrightarrows Z \times Z \times Z \cdots$$

The structure maps are

$$\begin{cases} d_i : Z_{n+1} \to Z_n & = & \text{projection by deleting the } i\text{th coordinate} \\ & & \text{e.g. } d_i(z_0, \ldots, z_n) = (z_0, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n) = \bar{z}_{\hat{i}} \\ & & \\ s_i : Z_n \to Z_{n+1} & = & \text{inclusion and diagonal applied to the } i\text{th coordinate} \\ & & \text{e.g. } s_i(z_0, \ldots, z_{n-1}) = (z_0, \ldots, z_{i-1}, z_i, z_i, z_{i+1}, \ldots, z_{n-1}) \end{cases}$$

Recall (from, e.g.[Dug08]) that for $\mathcal{Y}_{\bullet}$ a simplicial set augmented by the map $d_0 : \mathcal{Y}_0 \to *$ (i.e., $* =: \mathcal{Y}_{-1}$), a (forward) *contracting homotopy* of $\mathcal{Y}_{\bullet}$ is given by a collection of maps $\mathcal{S} : \mathcal{Y}_n \to \mathcal{Y}_{n+1}$ for $n \geq -1$ such that for each $y \in \mathcal{Y}_n$, one has

$$\begin{cases} d_i(\mathcal{S}y) & = & \begin{cases} \mathcal{S}(d_i y) & \text{if } 0 \leq i < n \\ y & \text{if } i = n \end{cases} \\ & & \\ \mathcal{S}(s_i y) & = & s_i(\mathcal{S}y) \qquad\qquad \text{for } 0 \leq i \leq n \end{cases}$$

First choose a point $v \in Z$. We set $\mathcal{S}(*)$(in our -1st dimension) to be $v \in Z$. For n-simplices z for $n > -1$, we define $\mathcal{S}(z) := (z, v)$. That is, if $z = (z_0, \ldots, z_n) \in \prod^{n+1} Z$, then $\mathcal{S}(z) = (z_0, \ldots, z_n, v)$. This is our desired contracting homotopy

□

4.1. **Contractibility of of $p \mapsto \mathrm{Top}(\mathscr{X}_k(p), Z)$.** The k-skeleton of a (co)simplicial object is adjoint to its k-coskeleton. With this, we have, for every j, the following isomorphisms of sets:

$$\begin{aligned}\mathrm{Top}(|\operatorname{sk}_k \Delta^j|, Z) &\cong \mathrm{sSets}(\operatorname{sk}_k \Delta^j, \mathrm{Sing}(Z)) \\ &\cong \mathrm{sSets}(\Delta^j, \operatorname{cosk}_k \mathrm{Sing}(Z)) \\ &:= \operatorname{cosk}_k(\mathrm{Sing}(Z))_j\end{aligned}$$

That is, $p \mapsto \mathrm{Top}(\mathscr{X}_k(p), Z) \cong \operatorname{cosk}_k(\mathrm{Sing}(Z))$.

For $Y_\bullet$ a simplicial set, we have that the map $Y_\bullet \to \operatorname{cosk}_k Y_\bullet$ is 1-1 and onto for dimensions $\leq k$, which implies that the homotopy groups of the two objects are the same in dimensions $< k$ (this is discussed in [DK84, §1.2,part (vi)]). We also have that the homotopy groups of $\operatorname{cosk}_k Y_\bullet$ are trivial in dimensions $\geq k$, when $Y_\bullet$ is fibrant.

Singularization produces fibrant simplicial sets, so we know that $\mathrm{Sing}(Z)$ is fibrant and therefore $\pi_i \operatorname{cosk}_k \mathrm{Sing}(Z) \cong 0$ for all $i \geq k$. We assumed that Z was $(k-1)$ connected, so for $i \leq (k-1)$, $\pi_i \operatorname{cosk}_k \mathrm{Sing}(Z) \cong \pi_i \operatorname{cosk}_k Z \cong 0$. We have just shown that all of its homotopy groups are trivial, i.e. it is weakly contractible. Its realization is a space of CW type, so by the Whitehead theorem, weakly contractible implies contractible. □

4.2. **Contractibility of of $p \mapsto \mathrm{Top}(\mathscr{Y}_k(p), Z)$.** This proof will be by induction. Note that the base case $k = 0$ is Lemma 4.1. We then assume that for all $0 < k < K$, $\mathrm{Top}(\mathscr{Y}_k(\bullet), Z)$ is contractible.

For the general case, we will express $\mathrm{Top}(\mathscr{Y}_K(\bullet), Z)$ as the homotopy pullback of contractible simplicial sets, and conclude that it is contractible.

A common model for the join of two spaces is the following pushout:

$$\begin{tikzcd} X \times Y \arrow[r] \arrow[d] & CX \times Y \arrow[d] \\ X \arrow[r] & X * Y \end{tikzcd}$$

Therefore, for all $j \geq 0$, $\mathscr{Y}_K(j) = \operatorname{sk}_0 \Delta^j * \mathscr{Y}_{K-1}(j)$ is the following pushout of spaces:

$$\begin{tikzcd} \mathscr{Y}_{K-1}(j) \times \operatorname{sk}_0 \Delta^j \arrow[r] \arrow[d] & C\mathscr{Y}_{K-1}(j) \times \operatorname{sk}_0 \Delta^j \arrow[d] \\ \mathscr{Y}_{K-1}(j) \arrow[r] & \mathscr{Y}_K(j) := \mathscr{Y}_{K-1}(j) * \operatorname{sk}_0 \Delta^j \end{tikzcd}$$

We then apply $\mathrm{Top}(\ , Z)$, the hom-set in topological spaces (of CW type), with Z $(k-1)$-connected. This functor takes pushouts to pullbacks (strict), and we have that the resultant square is, for each j, a pullback of sets:

$$\begin{tikzcd} \mathrm{Top}(\mathscr{Y}_{K-1}(j) * \operatorname{sk}_0 \Delta^j, Z) \arrow[r] \arrow[d] & \mathrm{Top}(\mathrm{C}(\mathscr{Y}_{K-1}(j)) \times \operatorname{sk}_0 \Delta^j, Z) \arrow[d] \\ \mathrm{Top}(\mathscr{Y}_{K-1}(j), Z) \arrow[r] & \mathrm{Top}(\mathscr{Y}_{K-1}(j) \times \operatorname{sk}_0 \Delta^j, Z) \end{tikzcd}$$

FIGURE 4. Levelwise pullback of sets

We will first show contractibility of the simplicial sets, then that the square is a homotopy pullback of simplicial sets (when j is allowed to vary).

By the induction hypothesis, we have that $\mathrm{Top}(\mathscr{Y}_{K-1}(\bullet), Z)$ is a contractible simplicial set. For the other two simplicial sets, if we allow our indices to vary independently, we may view our square as one of bisimplicial sets, and make use of results of Bousfield and Friedlander. Note that we consider $\mathrm{Top}(\mathscr{Y}_{K-1}(\bullet), Z)$ bisimplicial by making it constant in one direction. See Figure 5.

$$\begin{tikzcd}
\mathrm{Top}(\mathscr{Y}_{K-1}(i) * \mathrm{sk}_0\,\Delta^j, Z) \arrow[r] \arrow[d] & \mathrm{Top}(C(\mathscr{Y}_{K-1}(i) \times \mathrm{sk}_0\,\Delta^j, Z) \arrow[d] \\
\mathrm{Top}(\mathscr{Y}_{K-1}(i), Z) \arrow[r] & \mathrm{Top}(\mathscr{Y}_{K-1}(i) \times \mathrm{sk}_0\,\Delta^j, Z)
\end{tikzcd}$$

Figure 5. Square of bisimplicial sets (the indices $i, j \geq 0$ vary independently)

Using the following result, we can show levelwise equivalence (i.e. contractibility) and conclude that the diagonal is also contractible:

Theorem 4.2. [BF78, p119, Theorem B.2] *Let $f : X \to Y$ be a map of bisimplicial sets such that $f_{m,*} : X_{m,*} \to Y_{m,*}$ is a weak equivalence for each $m \geq 0$. Then $diag(f) : diagX \to diagY$ is a weak equivalence.*

Note that we have for each $m \geq 0$ the following isomorphisms:

$$\begin{array}{lcl}
\mathrm{Top}(C(\mathscr{Y}_{K-1}(m)) \times \mathrm{sk}_0\,\Delta^\bullet, Z) & \cong & \mathrm{Top}(\mathrm{sk}_0\,\Delta^\bullet, \mathrm{hom}_{\mathrm{Top}}(C(\mathscr{Y}_{K-1}(m)), Z)) \\
\mathrm{Top}(\mathscr{Y}_{K-1}(m) \times \mathrm{sk}_0\,\Delta^\bullet, Z) & \cong & \mathrm{Top}(\mathrm{sk}_0\,\Delta^\bullet, \mathrm{hom}_{\mathrm{Top}}(\mathscr{Y}_{K-1}(m), Z))
\end{array}$$

Then, by Lemma 4.1, these are contractible, since we have expressed them as $\mathrm{Top}(\mathrm{sk}_0\,\Delta^\bullet, X)$ for X a space. By Theorem 4.2, we may also conclude that $\mathrm{Top}(C(\mathscr{Y}_{K-1}(\bullet)) \times \mathrm{sk}_0\,\Delta^\bullet, Z)$ and $\mathrm{Top}(\mathscr{Y}_{K-1}(\bullet) \times \mathrm{sk}_0\,\Delta^\bullet, Z)$ are contractible, as we have shown levelwise contractibility.

To show that Figure 4 is a *homotopy* pullback, we continue to regard our square as one of bisimplicial sets, as in Figure 5, and use the following Theorem:

Theorem 4.3. [BF78, Theorem B.4] *Let*

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & X \arrow[d] \\
W \arrow[r] & Y
\end{tikzcd}$$

be a commutative square of bisimplicial sets such that the terms $V_{m,*}$, $W_{m,*}$, $X_{m,*}$ and $Y_{m,*}$ form a homotopy fiber square for each $m \geq 0$. If $X_{m,*}$ and $Y_{m,*}$ are all connected, then

$$\begin{tikzcd}
\mathrm{diag}V \arrow[r] \arrow[d] & \mathrm{diag}X \arrow[d] \\
\mathrm{diag}W \arrow[r] & \mathrm{diag}Y
\end{tikzcd}$$

is a homotopy fiber square.

First note that X and Y in our case have already been shown to be contractible, so are therefore connected. We will show that our bisimplicial set diagram levelwise is homotopy pullback squares of simplicial sets. That is, for all $m \geq 0$, the following is not just a levelwise pullback of sets but a homotopy pullback of simplicial sets:

$$\begin{tikzcd}
\mathrm{Top}(\mathscr{Y}_{K-1}(m) * \mathrm{sk}_0\,\Delta^\bullet, Z) \arrow[r] \arrow[d] & \mathrm{Top}(C(\mathscr{Y}_{K-1}(m)) \times \mathrm{sk}_0\,\Delta^\bullet, Z) \arrow[d] \\
\mathrm{Top}(\mathscr{Y}_{K-1}(m), Z) \arrow[r] & \mathrm{Top}(\mathscr{Y}_{K-1}(m) \times \mathrm{sk}_0\,\Delta^\bullet, Z)
\end{tikzcd}$$

We then show that the righthand vertical map is a Kan fibration[5], and conclude that our square is a homotopy pullback.

[5]We only need to show one map is a fibration due to right properness of the category of simplicial sets. See the gluing lemma, e.g. in [Sch97], Lemma 1.19.

Simplicial sets are a simplicial model category, satisfying Quillen's SM7 axiom, which is as follows[6]:

Quillen's SM7 axiom: *Let $\mathscr{C}$ be a simplicial model category, and $\mathrm{Hom}_{\mathscr{C}}(X, Z)$ denote the simplicial set of morphisms between X and Z. If $Y \in \mathscr{C}$ is fibrant and $f : A \to B$ is a cofibration in $\mathscr{C}$, then $\mathrm{Hom}_{\mathscr{C}}(f, Y) : \mathrm{Hom}_{\mathscr{C}}(B, Y) \to \mathrm{Hom}_{\mathscr{C}}(A, Y)$ is a fibration of simplicial sets.*

We apply several adjunctions to get the following isomorphisms. Note that $\mathrm{sk}_0\,\Delta^\bullet$ is equivalent to the levelwise simplicial set skeleton, i.e. $\mathrm{sk}_0\,\Delta^\bullet : n \mapsto \mathrm{sk}_0\,\Delta^n$, so it is adjoint levelwise to the simplicial coskeleton (for discussions of skeleta/coskeleta, see [GJ99, Ch. VII, mainly §1]).

$$\begin{aligned}
\mathrm{Top}(C\mathscr{Y}_{K-1}(m) \times \mathrm{sk}_0\,\Delta^\bullet, Z) &\cong \mathrm{sSet}(C\mathscr{Y}_{K-1}(m) \times \mathrm{sk}_0\,\Delta^\bullet, \mathrm{Sing}(Z)) \\
&\cong \mathrm{sSet}(C\mathscr{Y}_{K-1}(m), \mathrm{Hom}_{sSet}(\mathrm{sk}_0\,\Delta^\bullet, Z)) \\
&\cong \mathrm{sSet}(C\mathscr{Y}_{K-1}(m), \mathrm{Hom}_{sSet}(\Delta^\bullet, \mathrm{cosk}_0\,\mathrm{Sing}(Z))) \\
&\cong \mathrm{sSet}(C\mathscr{Y}_{K-1}(m) \times \Delta^\bullet, \mathrm{cosk}_0\,\mathrm{Sing}(Z)) \\
&=: \mathrm{Hom}_{sSet}(C\mathscr{Y}_{K-1}(m), \mathrm{cosk}_0\,\mathrm{Sing}(Z))
\end{aligned}$$

and likewise,

$$\begin{aligned}
\mathrm{Top}(\mathscr{Y}_{K-1}(m) \times \mathrm{sk}_0\,\Delta^\bullet, Z) &\cong \mathrm{sSet}(\mathscr{Y}_{K-1}(m) \times \mathrm{sk}_0\,\Delta^\bullet, \mathrm{Sing}(Z)) \\
&\cong \mathrm{sSet}(\mathscr{Y}_{K-1}(m) \times \Delta^\bullet, \mathrm{cosk}_0\,\mathrm{Sing}(Z)) \\
&=: \mathrm{Hom}_{sSet}(\mathscr{Y}_{K-1}(m), \mathrm{cosk}_0\,\mathrm{Sing}(Z))
\end{aligned}$$

We may express our righthand vertical map as

$$\mathrm{Hom}_{sSet}(C\mathscr{Y}_{K-1}(m), \mathrm{cosk}_0\,\mathrm{Sing}(Z)) \to \mathrm{Hom}_{sSet}(\mathscr{Y}_{K-1}(m), \mathrm{cosk}_0\,\mathrm{Sing}(Z))$$

which is $\mathrm{Hom}_{sSet}(\quad, \mathrm{cosk}_0\,\mathrm{Sing}(Z))$ applied to the map $\mathscr{Y}_{K-1}(m) \to C\mathscr{Y}_{K-1}(m)$. This map is a cofibration of simplicial sets since it is a monomorphism. The singularization of a topological space is a fibrant simplicial set, and as coskeleton is a right Kan extension, it preserves fibrant objects. Therefore, we may apply SM7 and conclude that our map is a fibration of simplicial sets, and our square is a homotopy pullback square. Applying Theorem 4.3, we conclude that Figure 4 is a homotopy pullback of simplicial sets which we have already shown to be contractible, so we conclude that $\mathrm{Top}(\mathscr{Y}_{K-1}(\bullet) * \mathrm{sk}_0\,\Delta^\bullet Z)$ is also a contractible simplicial set. ◻

References

[AK98] G. Arone and M. Kankaanrinta, A functorial model for iterated Snaith splitting with applications to calculus of functors, Stable and unstable homotopy (Toronto, ON, 1996) **19** (1998), 1–30.

[BEJM11] Kristine Bauer, Rosona Eldred, Brenda Johnson, and Randy McCarthy, Geometric models for Taylor polynomials of functors, In Preparation, 2011.

[BF78] A. Bousfield and E. Friedlander, Homotopy theory of γ-spaces, spectra, and bisimplicial sets, Geometric applications of homotopy theory II (1978), 80–130.

[BK72] A.K. Bousfield and D.M. Kan, Homotopy limits, completions and localizations, Springer, 1972.

[BM04] K. Bauer and R. McCarthy, On vanishing Tate cohomology and decompositions in Goodwillie calculus, preprint version (2004).

[Bro62] E.H. Brown, Cohomology theories, The Annals of Mathematics **75** (1962), no. 3, 467–484.

[Cha04] P. Chaoha, Splitting criteria for homotopy functors of spectra, TRANSACTIONS-AMERICAN MATHEMATICAL SOCIETY **356** (2004), no. 4, 1271–1280.

[DK84] WG Dwyer and DM Kan, An obstruction theory for diagrams of simplicial sets, Indagationes Mathematicae (Proceedings) **87** (1984), no. 2, 139–146.

[Dug08] D. Dugger, A primer on homotopy colimits, Preprint, 2008.

[ES52] S. Eilenberg and N.E. Steenrod, Foundations of algebraic topology, vol. 133, Princeton University Press, 1952.

[Goe93] P.G. Goerss, Barratt's desuspension spectral sequence and the lie ring analyzer, The Quarterly Journal of Mathematics **44** (1993), no. 1, 43.

[Goo90] T Goodwillie, Calculus I: The first derivative of pseudoisotopy theory, K-theory (1990), 1–27.

[Goo91] ———, Calculus II: analytic functors, K-theory (1991), 295–332.

[6] $\mathrm{Hom}_C(,)$ is the simplicial set from the simplicial model structure, $\mathrm{Hom}_C(X, Y) := C(X \times \Delta^\bullet, Y)$ where $C(X, Y)$ is the hom-set of morphisms in C.

[Goo03] ———, Calculus III: Taylor series, Geometry and Topology (2003), 645–711.
[Hir02] P.S. Hirschhorn, Model categories and their localizations, Mathematical Surveys and Monographs, vol. 99, AMS Bookstore, 2002.
[Hop84a] M J Hopkins, Formulations of cocategory and the iterated suspension, Astérisque **113-114** (1984), 212–226, Algebraic homotopy and local algebra (Luminy, 1982).
[Hop84b] ———, Some problems in topology, Ph.D. thesis, Oxford University, 1984.
[JM98] B. Johnson and R. McCarthy, Linearization, Dold-Puppe stabilization, and MacLane's Q-construction, Transactions of the American Mathematical Society **350** (1998), no. 4, 1555–1593.
[Joh95] B Johnson, The derivatives of homotopy theory, Transactions of the American Mathematical Society **347** (1995), no. 4, 1295–1321.
[Kuh04] N.J. Kuhn, Tate cohomology and periodic localization of polynomial functors, Inventiones mathematicae **157** (2004), no. 2, 345–370.
[MO02] Andrew Mauer-Oats, Goodwillie calculi, Ph.D. thesis, University of Illinois at Urbana-Champaign, 2002.
[Rez08] C. Rezk, A streamlined proof of Goodwillie's n-excisive approximation, Arxiv preprint arXiv:0812.1324 (2008).
[Shi96] B.E. Shipley, Convergence of the homology spectral sequence of a cosimplicial space, American Journal of Mathematics **118** (1996), no. 1, 179–207.
[Sin09] D.P. Sinha, The topology of spaces of knots: cosimplicial models, American Journal of Mathematics **131** (2009), no. 4, 945–980.
[Whi62] GW Whitehead, Generalized homology theories, Transactions of the American Mathematical Society **102** (1962), no. 2, 227–283.
[Sch97] S. Schwede, Spectra in model categories and applications to the algebraic cotangent complex, Journal of Pure and Applied Algebra, **120**(1997), no. 1, ,77–104.
[GJ99] P. G. Goerss and J.F. Jardine. Simplicial Homotopy Theory, volume 174 of Progress in Math. Birkhaeuser Verlag, 1999.